

Lecture 24 (April 25, 2016)

Singular Perturbation

up till now: state equations depend smoothly on ϵ

now: discontinuous dependence of system properties on ϵ .

Standard Singular Perturbation model: $\epsilon > 0$ small

$$\dot{x} = f(t, x, z, \epsilon)$$

$$\epsilon \dot{z} = g(t, x, z, \epsilon)$$

Note: setting $\epsilon=0$ causes a fundamental & abrupt change in the dynamic properties of the system, i.e., $0 = g(t, x, z, 0)$

Discontinuity of solutions can be avoided if analyzed in separate time scales: x : slow, z : fast.

Assume $f, g \in C^1$ in their arguments for $(t, x, z, \epsilon) \in [0, t] \times D_x \times D_z \times [0, \epsilon_0]$

Def. Standard form if $0 = g(t, x, z, 0)$ (*) has $K \geq 1$ isolated real roots $z = h_i(t, x)$, $i = 1, \dots, K$ for each $(t, x) \in [0, t] \times D_x$.

This ensures that a well-defined n -dimensional reduced model will correspond to each root of (*):

Slow model (**): $\dot{x} = f(t, x, h_i(t, x), 0)$: quasi steady state model
dropped $i \nearrow$ (z has rapidly converged to a root)

See section 11.1 for several examples in which small parameters & singular Perturbation model arise.

Time scale Properties of the standard model

slow response approximated by the reduced model

Discrepancy between the response of the reduced model and that of the full model is the fast transient.

Consider solving full model $\begin{cases} \dot{x} = f(t, x, z, \epsilon), & x(t_0) = \xi(\epsilon) \\ \epsilon \dot{z} = g(t, x, z, \epsilon), & z(t_0) = \eta(\epsilon) \end{cases}$

where $\xi(\epsilon)$ & $\eta(\epsilon)$ depend smoothly on ϵ and $t_0 \in [0, t_1]$.

Let $\underline{x}(t, \epsilon)$ & $\underline{z}(t, \epsilon)$ denote solutions to full model.

Reduced model $\dot{x} = f(t, x, h(t, x), 0), \quad x(t_0) = \xi_0 = \xi(0)$

Let $\bar{x}(t)$ be solution of reduced model with $\bar{z}(t) = h(t, \bar{x}(t))$
(quasi-steady-state behavior of z when $x = \bar{x}$)

$x(t_0, \epsilon) - \bar{x}(t_0) = \xi(\epsilon) - \xi(0) = O(\epsilon)$ (since ξ depends on ϵ smoothly)
so it is reasonable to expect $x(t, \epsilon) - \bar{x}(t) = O(\epsilon) \quad \forall t \in [t_0, t_1]$.

Note that $\bar{z}(t_0) = h(t_0, \bar{x}(t_0)) = h(t_0, \xi_0) \neq z(t_0) = \eta(\epsilon)$
not necessarily

Thus, best we can expect is that $z(t, \epsilon) - \bar{z}(t) = O(\epsilon)$ on an interval excluding t_0 , i.e., for $t \in [t_b, t_1]$ where $t_b > t_0$.

If error $z(t, \epsilon) - \bar{z}(t)$ is $O(\epsilon)$ on $[t_b, t_1]$, then during $[t_0, t_b]$ (called the "boundary-layer" interval), z approaches \bar{z} .

Need some stability in order for this to happen.

Let $y = z - h(t, x)$, so quasi-steady-state shifts to origin in y coordinate $y=0$. This change of variable doesn't change the equation of reduced model.

(Note that if we find an equation in terms of y and approximate its solution (say $\hat{y}(t)$) then $z \approx \hat{y}(t) + h(t, x)$.)

Let $\tau = \frac{t-t_0}{\epsilon}$. So $\tau=0$ when $t=t_0$ and time is stretched for $\epsilon \ll 1$.

$$\begin{cases} \epsilon \frac{dy}{dt} = \frac{dy}{dz} = g(t, x, y + h(t, x), \epsilon) - \epsilon \frac{\partial h}{\partial t} - \epsilon \frac{\partial h}{\partial x} f(t, x, y + h(t, x), \epsilon) \\ y(0) = \eta(\epsilon) - h(t_0, \xi(\epsilon)) \end{cases}$$

Note that t & x are slowly varying in the τ time scale:

$$t = t_0 + \epsilon \tau, \quad x = x(t_0 + \epsilon \tau, \epsilon)$$

Setting $\epsilon=0$ freezes these at $t=t_0$ and $x=\xi_0$, which gives:

$$\text{(†)} \quad \begin{cases} \frac{dy}{d\tau} = g(t_0, \xi_0, y + h(t_0, \xi_0), 0) \\ y(0) = \eta_0 - h(t_0, \xi_0) \end{cases}$$

which has an eq. pt at $y=0$.

If $y=0$ is a.s. and $y(0)$ is in its region of attraction, then it is reasonable that solution will reach $O(\epsilon)$ neighborhood of origin during boundary-layer interval. (In non-stretched time)

Still need to guarantee that it stays there for $t \in [t_b, t_f]$ since in the latter interval, t, x , move away from their initial values. So let them differ.

$$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0) \quad \text{Boundary-layer-model (BLM)}$$

where $(t, x) \in [0, t_f] \times D_x$ treated as fixed parameters, and $\bar{x}(t) \in D_{\bar{x}}$.

The crucial stability property we need for (BLM) is e.s.-ity of its origin, uniformly in the frozen parameters.

(If $y=0$ is e.s. for (†), then it remains e.s. for BLM.)

Tikhonov's Theorem

consider the singular perturbation problem

$$\begin{cases} \dot{x} = f(t, x, z, \epsilon), \quad x(t_0) = \xi(\epsilon) \\ \epsilon \dot{z} = g(t, x, z, \epsilon), \quad z(t_0) = \eta(\epsilon) \end{cases}$$

and let $z = h(t, x)$ be an isolated root of $g(t, x, z, 0) = 0$. Assume the following conditions for all $[t, x, z - h(t, x), \varepsilon] \in [0, t_1] \times D_x \times D_y \times [0, \varepsilon_0]$

$$Dx \subset \mathbb{R}^n \text{ convex} \quad D_y \subset \mathbb{R}^m \text{ contains } 0$$

- f, g , first partial derivatives w.r.t (y, z, ε) & first partial derivative of g w.r.t t are continuous.
- h & $\frac{\partial g}{\partial z}(t, x, z, 0)$ have continuous first partial derivatives w.r.t their arguments.
- $\xi(\varepsilon)$ & $\eta(\varepsilon)$ are smooth functions of ε .
- The reduced problem has a unique solution $\bar{x}(t) \in S$, $\forall t \in [t_0, t_1]$
 $S \subset D_x$ compact
- The origin is e.s. eq. pt of (BLM), uniformly in (t, x)
- Let $R_y \subset D_y$: region of attraction of $(\#)$ & $\Omega_y \subset R_y$ compact
 Then, $\exists \varepsilon^* > 0$ s.t. $\forall y(0) = \eta_0 - h(t_0, \xi_0) \in \Omega_y$ and $0 < \varepsilon < \varepsilon^*$, the singular perturbation problem has a unique solution $x(t, \varepsilon)$ & $z(t, \varepsilon)$ on $[t_0, t_1]$ and

$$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon)$$

$$z(t, \varepsilon) - \hat{y}(t, \varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon)$$

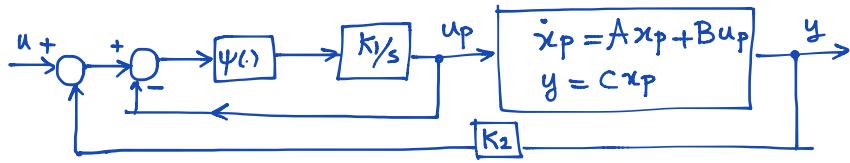
hold uniformly for $t \in [t_0, t_1]$, where $\hat{y}(t)$ is the solution of $(\#)$.

Moreover, given any $t_b > t_0$, $\exists \varepsilon^{**} < \varepsilon^*$ s.t. $z(t, \varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon)$ holds uniformly for $t \in [t_b, t_1]$, $\varepsilon < \varepsilon^{**}$.

Main idea of proof. Showing $\|y(t, \varepsilon)\| \leq k_1 e^{-\alpha(t-t_0)/\varepsilon} + \varepsilon \delta$ using the stability properties of (LBM).

Note. To get extension to infinite time interval, we also require e.s.-ity of origin of reduced model. (see Theorem II.2).

Example 11.2 & 11.5 (High gain control)



Actuator control with high-gain feedback

Inner loop: actuator control with high gain feedback, K_1 : high-gain, $\psi(\cdot) \in (0, \infty]$
 $\psi(0)=0$, $y \psi(y) > 0 \forall y \neq 0$.

state equation: $\dot{x}_p = Ax_p + Bu_p$

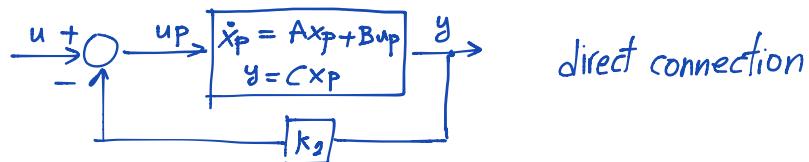
$$\frac{1}{K_1} \dot{u}_p = \psi(u - u_p - K_2 C x_p)$$

Let $\epsilon = \frac{1}{K_1}$, $x_p = x$, $u_p = z$, get singular perturbation model

setting $\epsilon=0 \Rightarrow \psi(u - u_p - K_2 C x_p) = 0 \Rightarrow u_p = u - K_2 C x_p$

This is a unique root since ψ vanishes only at origin.

Reduced model: $\dot{x}_p = (A - B K_2 C) x_p + B u$



Suppose $u(t)=1$ for $t \geq 0$ and $\psi(\cdot) = \tan^{-1}(\cdot)$. The unique root of $\psi(u - z - K_2 C x) = 0$ is $h(t, x) = 1 - K_2 C x$ and the (BLM) is

$$\begin{aligned} \frac{dy}{dt} &= g(t, z, y + h(t, x), 0) = \psi(1 - (y + h(t, x)) - K_2 C x) \\ &= \psi(1 - y - 1 + K_2 C x - K_2 C x) \\ &= \tan^{-1}(-y) = -\tan^{-1}y \end{aligned}$$

The Jacobian $\frac{\partial g}{\partial y} \Big|_{y=0} = -\frac{1}{1+y^2} \Big|_{y=0} = -1$ is Hurwitz \Rightarrow

The origin of (BLM) is e.s.

one can check that all assumptions of Theorem II.1 are satisfied, and can proceed to approximate x by the solution of the reduce model, and z by $h(t, \bar{x}(t)) + \hat{y}(t/\epsilon)$

the solution of reduced model the solution of $\frac{dy}{\tau} = -\tan^{-1}(y)$

$\hat{y}(t/\epsilon) = \sec^{-1}(c e^{-t/\epsilon})$ corresponds to fast transient.
constant

(Boundary-layer part of the solution)

After the decay of this transient, z remains close to $h(t, \bar{x}(t))$, which is slow(quasi steady state) part of the solution.

Do Example II.5 if you have time.

HW #5 9.11 / 10.15 (1-2) / 11.10

